Definition

- A set $N \subset \mathbb{R}$ is called a null set if for any $\varepsilon > 0$, there exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals such that $N \subset \bigcup_{k=1}^{\infty} (a_k, b_k)$ and $\sum_{k=1}^{\infty} (b_k - a_k) < \varepsilon$.
- A property (P) holds almost everywhere if and only if (P) holds for all $x \in \mathbb{R} \setminus N$ where N is a null set.

Rmk: We can replace the open intervals in the definition by closed intervals since $(a_k, b_k) \subset [a_k, b_k] \subset (a_k - \frac{\varepsilon}{2^k}, b_k + \frac{\varepsilon}{2^k})$.

Example

Any countable set is a null set.

Pf: Write $E = \{x_k\}_{k=1}^{\infty}$.

Take $(a_k, b_k) = (x_k - \frac{\varepsilon'}{2^{k+1}}, x_k + \frac{\varepsilon'}{2^{k+1}})$ with $\varepsilon' < \varepsilon$.

Then $b_k - a_k = \frac{\varepsilon'}{2^k}$.

Thus $\sum_{k=1}^{\infty} (b_k - a_k) = \sum_{k=1}^{\infty} \frac{\varepsilon'}{2^k} = \varepsilon' < \varepsilon$.

Proposition

Let f be a nonnegative real function on $[a,b]$.

If $f \in R[a,b]$ and $\int_a^b f = 0$, then $f=0$ almost everywhere.

Lemma

A countable union of null sets is still a null set.

Pf: Let $(E_n)_{n=1}^{\infty}$ be a countable collection of null sets.

Fix $\varepsilon > 0$.

For any $n \in \mathbb{N}$, since E_n is a null set, there exists a countable collection $(I_{n,k})_{k=1}^{\infty}$ of open intervals such that $E_n \subset \bigcup_{k=1}^{\infty} I_{n,k}$ and $\sum_{k=1}^{\infty} |I_{n,k}| < \frac{\varepsilon}{2^n}$.

Now consider the collection of intervals $(I_{n,k})_{k,n=1}^{\infty}$. It is still a countable collection. (1050 Exercise!)

Moreover, $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{n,k}$ and

$$\sum_{n,k=1}^{\infty} |I_{n,k}| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |I_{n,k}| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon .$$

$\square$

Pf of proposition:

We wish to show $\{x \in [a,b] : f(x) \neq 0\}$ is a null set. Since f is nonnegative, it is equivalent to show $\{x \in [a,b] : f(x) > 0\}$ is a null set.

Note that $\{x \in [a,b] : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in [a,b] : f(x) > \frac{1}{n}\}$ (AP)

By Lemma, it suffices to show for any $n \in \mathbb{N}$,

$\{x \in [a,b] : f(x) > \frac{1}{n}\}$ is a null set.

Actually, we can show $E = \{x \in [a,b] : f(x) > c\}$ is a null set for any $c > 0$.

Suppose E is not a null set.

Then there exists $\varepsilon_0 > 0$ such that for any countable collection $(J_k)_{k=1}^{\infty}$ of closed intervals

by Rmk below Def

satisfying $E \subset \bigcup_{k=1}^{\infty} J_k$, we have $\sum_{k=1}^{\infty} |J_k| > \varepsilon_0$.

Pick any partition $P = (x_0, x_1, \dots, x_n)$ of $[a,b]$

Let $I_k = [x_{k-1}, x_k]$

Since $\bigcup_{k=1}^{n} I_k = [a,b]$, $\bigcup_{k: I_k \cap E \neq \phi} I_k \supset E$.

Then $\sum_{k: I_k \cap E \neq \phi} |I_k| > \varepsilon_0$

$$U(f;P) = \sum_{k=1}^{n} \sup\{f(x) : x \in [x_{k-1}, x_k]\} (x_k - x_{k-1})$$

$$= \sum_{k: I_k \cap E = \phi} \sup\{f(x) : x \in [x_{k-1}, x_k]\} (x_k - x_{k-1})$$

$$+ \sum_{k: I_k \cap E \neq \phi} \sup\{f(x) : x \in [x_{k-1}, x_k]\} (x_k - x_{k-1})$$

(f non-negative) $$\geq \sum_{k: I_k \cap E \neq \phi} \sup\{f(x) : x \in [x_{k-1}, x_k]\} (x_k - x_{k-1})$$

(def of E) $$\geq c \sum_{k: I_k \cap E \neq \phi} (x_k - x_{k-1})$$

$$> c\varepsilon_0 > 0.$$

Since for any partition P of $[a,b]$

$U(f;P) > c\varepsilon_0$ where c, ε_0 are independent of P,

then $U(f) > c\varepsilon_0 > 0$.

But $U(f) = \int_a^b f = 0$

Contradiction!

Let f be nonnegative bounded real function.

Is $\int_a^b f = 0$ equivalent to $f = 0$ a.e. ?

In Riemann Integral setting, the answer is no.

Example: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0, & x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$

Since $\mathbb{Q}$ is countable, thus a null set,

$f = 0$ almost everywhere.

But $U(f) = 1 \neq 0 = L(f)$ by density of

$\mathbb{Q}$ and $\mathbb{Q}^c$. Therefore, $f \notin R[0,1]$.

However, in Lebesgue Integral setting,

$$\int_a^b f = 0 \iff f = 0 \quad \text{a.e.}$$

for nonnegative f.

This is a reason for the development of Lebesgue Integral.